Elliptic Curves and Modular Forms
Final Examination

November 18 2016

This exam is of 50 marks. Please read all the questions carefully and do not cheat. You are allowed to use

- *Silverman and Tate - Rational Points on Elliptic Curves.*
- *Ireland and Rosen - A Classical Introduction to Modern Number Theory.*
- *Serre - A Course in Arithmetic*
- *Koblitz - Introduction to Elliptic Curves and Modular Forms*

Please feel free to use whatever theorems you have learned in class after stating them clearly. We will use Koblitz's notation of weight of the modular form being k as opposed to Serre's $2k$.

1. Consider the elliptic curve

$$E : y^2 = x^3 - 43x + 166$$

Compute the **group of torsion points** in $E(\mathbb{Q})$. (10)

2a. Let p be a prime and $d = (m, p-1)$. Prove that for any $a \bmod p$,

$$N(x^m = a) = N(x^d = a)$$

where $N(x^m = a)$ is the number of solutions of the equation $\bmod p$. (5)

2b. Count the number of solutions of (5)

$$x^9 + y^9 = 1$$

in the finite field $\mathbb{F}_{31}$.

3a. Let f be a modular form of weight k for $\Gamma = SL_2(\mathbb{Z})$. Show that (7)

$$g(z) = \frac{1}{2\pi i} f'(z) - \frac{k}{12} E_2(z) f(z)$$

is a modular form of weight $k+2$. Here $E_2(z)$ is the non-modular Eisenstein series of weight 2 given by

$$E_2(z) = \frac{1}{2\zeta(2)} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{1}{(mz+n)^2}$$

where $'$ means $(m, n) \neq (0, 0)$.

b. Show that g is a cusp form if and only if f is one as well. (3)

4a. What is the order of the group $SL_2(\mathbb{Z}/6\mathbb{Z})$? (5)

4b. Find the cardinality of the coset space $\Gamma_1(6)/\Gamma(6)$. (5)

5. Let $\mu(n)$ be the Möbius function defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \text{ is not square-free} \\ (-1)^r & \text{if } n = p_1 \dots p_r \end{cases}$$

a. Show that μ is multiplicative: $\mu(mn) = \mu(m)\mu(n)$ if $(n, m) = 1$. (3)

b. Compute, for $\mathrm{Re}(s) > 1$, (4)

$$G(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

in terms of the Riemann ζ-function.

c. Use what you know about the values of the Riemann ζ-function to compute the value $G(2)$. (3)